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LETTER TO THE EDITOR

Magnetic correlation length and universal amplitude of the lattice E_8 Ising model

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Abstract. The perturbation approach is used to derive the exact correlation length ξ of the dilute A_L lattice models in regimes 1 and 2 for L odd. In regime 2 the A_3 model is the E_8 lattice realization of the two-dimensional Ising model in a magnetic field h at $T = T_c$. When combined with the singular part f_s of the free energy the result for the A_3 model gives the universal amplitude $f_s\xi^2 = 0.061\,728\ldots$ as $h \to 0$ in precise agreement with the result obtained by Delfino and Mussardo via the form-factor bootstrap approach.

The integrable E_8 quantum field theory of Zamolodchikov [1, 2] is known to be in the same universality class as the two-dimensional Ising model in a magnetic field at $T = T_c$. Moreover, an integrable lattice realization of the E_8 Ising model is provided by the dilute A_3 model [3, 4], upon which explicit exact and numerical calculations pertaining to the Ising model in a magnetic field can be performed [3–13].

In this letter we present the correlation length of the dilute A_L lattice models in regimes 1 and 2 for L odd, for which the off-critical perturbation is magnetic-like. This includes the magnetic correlation length for $L = 3$, of relevance to the magnetic Ising model at $T = T_c$.

The dilute A_L model is an exactly solvable, restricted solid-on-solid model defined on the square lattice. Each site of the lattice can take one of L possible (height) values, subject to the restriction that neighbouring sites of the lattice either have the same height, or differ by ± 1. The Boltzmann weights of the allowed height configurations of an elementary face of the lattice are [3, 4]

$$W\begin{pmatrix} a & a \\ a & a \end{pmatrix} = \frac{\vartheta_1(6\lambda - u)\vartheta_1(3\lambda + u)}{\vartheta_1(6\lambda)\vartheta_1(3\lambda)} - \left(\frac{S(a+1)}{S(a)} \frac{\vartheta_4(2a\lambda - 5\lambda)}{\vartheta_4(2a\lambda + \lambda)} + \frac{S(a-1)}{S(a)} \frac{\vartheta_4(2a\lambda + 5\lambda)}{\vartheta_4(2a\lambda - \lambda)} \right) \frac{\vartheta_1(u)\vartheta_1(3\lambda - u)}{\vartheta_1(6\lambda)\vartheta_1(3\lambda)}$$

$$W\begin{pmatrix} a\pm 1 & a \\ a & a \end{pmatrix} = W\begin{pmatrix} a & a \\ a & a\pm 1 \end{pmatrix} = \frac{\vartheta_1(3\lambda - u)\vartheta_4(\pm 2a\lambda + \lambda - u)}{\vartheta_1(3\lambda)\vartheta_4(\pm 2a\lambda + \lambda)}$$

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$$W\begin{pmatrix} a & a \\ a\pm 1 & a\end{pmatrix} = W\begin{pmatrix} a & a\pm 1 \\ a & a\end{pmatrix} = \left(\frac{S(a\pm 1)}{S(a)}\right)^{1/2} \frac{\vartheta_1(u)\vartheta_4(\pm 2a\lambda - 2\lambda + u)}{\vartheta_1(3\lambda)\vartheta_4(\pm 2a\lambda + \lambda)}$$

$$W\begin{pmatrix} a & a\pm 1 \\ a & a\pm 1\end{pmatrix} = W\begin{pmatrix} a\pm 1 & a\pm 1 \\ a & a\end{pmatrix}$$

$$= \left(\frac{\vartheta_4(\pm 2a\lambda + 3\lambda)\vartheta_4(\pm 2a\lambda - \lambda)}{\vartheta_4^2(\pm 2a\lambda + \lambda)}\right)^{1/2} \frac{\vartheta_1(u)\vartheta_1(3\lambda - u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)}$$

$$W\begin{pmatrix} a\pm 1 & a \\ a & a\mp 1\end{pmatrix} = \frac{\vartheta_1(2\lambda - u)\vartheta_1(3\lambda - u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)}$$

$$W\begin{pmatrix} a & a\mp 1 \\ a\pm 1 & a\end{pmatrix} = -\left(\frac{S(a-1)S(a+1)}{S^2(a)}\right)^{1/2} \frac{\vartheta_1(u)\vartheta_1(\lambda - u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)}$$

$$W\begin{pmatrix} a & a\pm 1 \\ a\pm 1 & a\end{pmatrix} = \frac{\vartheta_1(3\lambda - u)\vartheta_1(\pm 4a\lambda + 2\lambda + u)}{\vartheta_1(3\lambda)\vartheta_1(\pm 4a\lambda + 2\lambda)}$$

$$+\frac{S(a\pm 1)}{S(a)} \frac{\vartheta_1(u)\vartheta_1(\pm 4a\lambda - \lambda + u)}{\vartheta_1(3\lambda)\vartheta_1(\pm 4a\lambda + 2\lambda)} = \frac{\vartheta_1(3\lambda + u)\vartheta_1(\pm 4a\lambda - 4\lambda + u)}{\vartheta_1(3\lambda)\vartheta_1(\pm 4a\lambda - 4\lambda)}$$

$$+\left(\frac{S(a\mp 1)}{S(a)}\frac{\vartheta_1(4\lambda)}{\vartheta_1(2\lambda)} - \frac{\vartheta_4(\pm 2a\lambda - 5\lambda)}{\vartheta_4(\pm 2a\lambda + \lambda)}\right) \frac{\vartheta_1(u)\vartheta_1(\pm 4a\lambda - \lambda + u)}{\vartheta_1(3\lambda)\vartheta_1(\pm 4a\lambda - 4\lambda)}. \qquad (1)$$

The crossing factors $S(a)$ are defined by

$$S(a) = (-1)^a \frac{\vartheta_1(4a\lambda)}{\vartheta_4(2a\lambda)} \qquad (2)$$

and $\vartheta_1(u)$, $\vartheta_4(u)$ are standard elliptic theta functions of nome p

$$\vartheta_1(u) = \vartheta_1(u, p) = 2p^{1/4}\sin u \prod_{n=1}^{\infty}(1 - 2p^{2n}\cos 2u + p^{4n})(1 - p^{2n}) \qquad (3)$$

$$\vartheta_4(u) = \vartheta_4(u, p) = \prod_{n=1}^{\infty}(1 - 2p^{2n-1}\cos 2u + p^{4n-2})(1 - p^{2n}). \qquad (4)$$

In the above weights the variable λ and the range of the spectral parameter u are given by $0 < u < 3\lambda$ with

$$\lambda = \frac{s}{r}\pi \qquad (5)$$

where $r = 4(L + 1)$ and $s = L$ in regime 1 and $s = L + 2$ in regime 2†. The magnetic Ising point occurs in regime 2 with $\lambda = 5\pi/16$.

The row transfer matrix of the dilute A models is defined on a periodic strip of width N as

$$T_{\{a\}}^{\{b\}} = \prod_{j=1}^{N} W\begin{pmatrix} b_j & b_{j+1} \\ a_j & a_{j+1}\end{pmatrix} \qquad (6)$$

where $\{a\}$ is an admissible path of heights and $a_{N+1} = a_1$, $b_{N+1} = b_1$. For convenience we take N even.

The eigenvalues of the transfer matrix are [6, 12, 13]

$$\Lambda(u) = \omega\left[\frac{\vartheta_1(2\lambda - u)\vartheta_1(3\lambda - u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)}\right]^N \prod_{j=1}^{N} \frac{\vartheta_1(u - u_j + \lambda)}{\vartheta_1(u - u_j - \lambda)}$$

† The model has other regimes, but they are not of interest here.

$$+\left[\frac{\vartheta_1(u)\vartheta_1(3\lambda-u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)}\right]^N\prod_{j=1}^{N}\frac{\vartheta_1(u-u_j)\vartheta_1(u-u_j-3\lambda)}{\vartheta_1(u-u_j-\lambda)\vartheta_1(u-u_j-2\lambda)}$$

$$+\omega^{-1}\left[\frac{\vartheta_1(u)\vartheta_1(\lambda-u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)}\right]^N\prod_{j=1}^{N}\frac{\vartheta_1(u-u_j-4\lambda)}{\vartheta_1(u-u_j-2\lambda)} \tag{7}$$

where the N roots u_j are given by the Bethe equations

$$\omega\left[\frac{\vartheta_1(\lambda-u_j)}{\vartheta_1(\lambda+u_j)}\right]^N=-\prod_{k=1}^{N}\frac{\vartheta_1(u_j-u_k-2\lambda)\vartheta_1(u_j-u_k+\lambda)}{\vartheta_1(u_j-u_k+2\lambda)\vartheta_1(u_j-u_k-\lambda)} \tag{8}$$

and $\omega=\exp(\mathrm{i}\pi\ell/(L+1))$ for $\ell=1,\ldots,L$.

There are several methods at hand to calculate the correlation length. Here we apply the perturbative approach initiated by Baxter [14, 15]. For L odd this involves perturbing away from the strong magnetic field limit at $p=1$. We thus introduce the variables

$$w=\mathrm{e}^{-2\pi u/\epsilon} \qquad \text{and} \qquad x=\mathrm{e}^{-\pi^2/r\epsilon} \tag{9}$$

conjugate to the nome $p=\mathrm{e}^{-\epsilon}$. The relevant conjugate modulus transformations are

$$\vartheta_1(u,p)=\left(\frac{\pi}{\epsilon}\right)^{1/2}\mathrm{e}^{-(u-\pi/2)^2/\epsilon}E(w,q^2) \tag{10}$$

$$\vartheta_4(u,p)=\left(\frac{\pi}{\epsilon}\right)^{1/2}\mathrm{e}^{-(u-\pi/2)^2/\epsilon}E(-w,q^2) \tag{11}$$

where $q=\mathrm{e}^{-\pi^2/\epsilon}$ and

$$E(z,p)=\prod_{n=1}^{\infty}(1-p^{n-1}z)(1-p^nz^{-1})(1-p^n). \tag{12}$$

In the ordered limit ($p\to 1$ with u/ϵ fixed) the Boltzmann weights for L odd reduce to

$$W\begin{pmatrix} d & c\\ a & b\end{pmatrix}\sim w^{H(d,a,b)}\delta_{a,c}. \tag{13}$$

The function $H(d,a,b)$ is given explicity in [5], being required for the calculation of the local height probabilities. In this limit the row transfer matrix eigenspectra breaks up into a number of bands labelled by integer powers of w. In regime 1 there are $\frac{1}{2}(L+1)$ ground states and in regime 2 there are $\frac{1}{2}(L-1)$ ground states, each with eigenvalue $\lambda_0=1$. The bands of excitations are relevant to the calculation of the correlation length.

The number of states in the w band is $\frac{1}{2}(L-1)N$ in regime 1 and $\frac{1}{2}(L-3)N$ in regime 2. These correspond to introducing in all but one of the ground-state paths $\{a\}$ a single non-ground-state height, in any position. In particular, note that there are *no* excitations in the w band for $L=3$ in regime 2. Thus for the magnetic Ising model we must consider excitations in the w^2 band. These are harder to count, arising from a variety of both single and multiple deviations from ground-state paths. However, we observe numerically that (apart from when $N=2$) there are $4N$ states in the w^2 band.

We associate a given value of ℓ with each eigenvalue by numerically comparing the eigenspectrum at criticality ($p=0$) with the eigenspectrum of the corresponding O(n) loop model [18] for finite N†. Each eigenvalue can then be tracked to the ordered limit. The band of largest eigenvalues is seen to have the values $\ell=1,\ldots,\frac{1}{2}(L+1)$ in regime 1 and $\ell=1,\ldots,\frac{1}{2}(L-1)$ in regime 2.

† Strictly speaking we compare with the eigenspectrum of the corresponding vertex model with seam ω.

Setting $w_j = \mathrm{e}^{-2\pi u_j/\epsilon}$, the eigenvalues (7) can be written

$$\Lambda(w) = \omega\left[\frac{E(x^{4s}/w, x^{2r})E(x^{6s}/w, x^{2r})}{E(x^{4s}, x^{2r})E(x^{6s}, x^{2r})}\right]^N \prod_{j=1}^{N} w_j^{1-2s/r}\frac{E(x^{2s}w/w_j, x^{2r})}{E(x^{2s}w_j/w, x^{2r})}$$
$$+\left[\frac{x^{2s}}{w}\frac{E(w, x^{2r})E(x^{6s}/w, x^{2r})}{E(x^{4s}, x^{2r})E(x^{6s}, x^{2r})}\right]^N \prod_{j=1}^{N} w_j \frac{E(w/w_j, x^{2r})E(x^{6s}w_j/w, x^{2r})}{E(x^{2s}w_j/w, x^{2r})E(x^{4s}w_j/w, x^{2r})}$$
$$+\omega^{-1}\left[x^{2s}\frac{E(w, x^{2r})E(x^{2s}/w, x^{2r})}{E(x^{4s}, x^{2r})E(x^{6s}, x^{2r})}\right]^N \prod_{j=1}^{N} w_j^{2s/r}\frac{E(x^{8s}w_j/w, x^{2r})}{E(x^{4s}w_j/w, x^{2r})}. \qquad (14)$$

The Bethe equations (8) are now

$$\omega\left[w_j\frac{E(x^{2s}/w_j, x^{2r})}{E(x^{2s}w_j, x^{2r})}\right]^N = -\prod_{k=1}^{N} w_k^{2s/r}\frac{E(x^{2s}w_j/w_k, x^{2r})E(x^{4s}w_k/w_j, x^{2r})}{E(x^{2s}w_k/w_j, x^{2r})E(x^{4s}w_j/w_k, x^{2r})}. \qquad (15)$$

The calculation of the largest eigenvalue proceeds from the $x \to 0$ limit with w fixed in a similar manner to that for the eight-vertex [16] and CSOS [17] models. Each of the degenerate ground states has a different root distribution $\{w_j\}$ on the unit circle, depending on ℓ. Defining the free energy per site as $f = N^{-1}\log\Lambda_0$, our final result is

$$f = 4\sum_{k=1}^{\infty}\frac{\cosh[(5\lambda-\pi)\pi k/\epsilon]\cosh(\pi\lambda k/\epsilon)\sinh(\pi u k/\epsilon)\sinh[(3\lambda-u)\pi k/\epsilon]}{k\ \sinh(\pi^2 k/\epsilon)\cosh(3\pi\lambda k/\epsilon)} \qquad (16)$$

in agreement with the previous calculations via the inversion relation method [3–5].

In regime 1, the leading eigenvalue in the w band has $\ell = \frac{1}{2}(L+1)+1$. The root distribution has $N-1$ roots on the unit circle and a 1-string excitation located exactly at $w_N = -x^r$. Applying perturbative arguments along the lines of [17] yields the leading excitation in the w band to be

$$\frac{\Lambda_1}{\Lambda_0} = w\frac{E(-x^{2s}/w, x^{12s})E(-x^{4s}/w, x^{12s})}{E(-x^{2s}w, x^{12s})E(-x^{4s}w, x^{12s})}. \qquad (17)$$

At the isotropic point $w = x^{3s}$ this reduces to

$$\frac{\Lambda_1}{\Lambda_0} = x^s\frac{E^2(-x^s, x^{12s})}{E^2(-x^{5s}, x^{12s})} = \left[\frac{\vartheta_4(\pi/12, p^{\pi/6\lambda})}{\vartheta_4(5\pi/12, p^{\pi/6\lambda})}\right]^2. \qquad (18)$$

For $L = 3$ in regime 2 extensive numerical investigations of the Bethe equations have led to a convincing conjecture for the thermodynamically significant strings [6, 8]. We find that the leading excitation in the w^2 band is a 2-string with $\ell = 2$. However, the state is originally a 1-string for small p. Such behaviour has been discussed in [9]. Tracking this state with increasing p reveals that the 2-string is exactly located at $-x^{\pm 11}$ in the limit $p = 1$. There are finite-size deviations away from this position for small N and $0 < p < 1$. The location we find for this string is in accord with the previous numerical work [6, 8]. Applying the perturbation arguments in this case yields the leading excitation in the w^2 band for $L = 3$ to be

$$\frac{\Lambda_2}{\Lambda_0} = w^2\frac{E(-x/w, x^{60})E(-x^{11}/w, x^{60})E(-x^{31}w, x^{60})E(-x^{41}w, x^{60})}{E(-xw, x^{60})E(-x^{11}w, x^{60})E(-x^{31}/w, x^{60})E(-x^{41}/w, x^{60})}. \qquad (19)$$

At the isotropic point $w = x^{15}$ this reduces to

$$\frac{\Lambda_2}{\Lambda_0} = x^{28}\frac{E^2(-x^4, x^{60})E^2(-x^{14}, x^{60})}{E^2(-x^{16}, x^{60})E^2(-x^{26}, x^{60})} = \left[\frac{\vartheta_4(\pi/15, p^{8/15})\vartheta_4(7\pi/30, p^{8/15})}{\vartheta_4(4\pi/15, p^{8/15})\vartheta_4(13\pi/30, p^{8/15})}\right]^2. \qquad (20)$$

The correlation length ξ can be obtained either by integrating over the relevant band of eigenvalues or via the leading eigenvalue in the band at the isotropic point (see, e.g., [17]). Doing the latter we have

$$\xi^{-1} = -\log \frac{\Lambda}{\Lambda_0} \tag{21}$$

where Λ is the relevant leading eigenvalue. Our final results are thus

$$\xi^{-1} = 2\log\left[\frac{\vartheta_4(5\pi/12,\, p^{\pi/6\lambda})}{\vartheta_4(\pi/12,\, p^{\pi/6\lambda})}\right] \tag{22}$$

for L odd in regime 1, with

$$\xi^{-1} = 2\log\left[\frac{\vartheta_4(4\pi/15,\, p^{8/15})\vartheta_4(13\pi/30,\, p^{8/15})}{\vartheta_4(\pi/15,\, p^{8/15})\vartheta_4(7\pi/30,\, p^{8/15})}\right] \tag{23}$$

for $L = 3$ in regime 2.

The derivation of the correlation length for $L \neq 3$ in regime 2 is complicated. In this regime the leading excitation in the w band has $\ell = \frac{1}{2}(L-1)+1$ and, like the leading 2-string in the w^2 band for $L = 3$, it begins life for small N and $p \simeq 0$ as a 1-string. We have not pursued this further. Nevertheless, we have numerically observed that the final result (17) also applies to the leading w band excitation in regime 2. We thus believe that the correlation length (22) and the corresponding exponents (25) below also hold in regime 2 for $L \neq 3$.

It follows from (22) that the correlation length diverges at criticality as

$$\xi \sim \frac{1}{4\sqrt{3}}\, p^{-\nu_h} \qquad \text{as } p \to 0 \tag{24}$$

where the correlation length exponent ν_h is given by

$$\nu_h = \frac{r}{6s} = \begin{cases} \dfrac{2(L+1)}{3L} & \text{regime 1} \\[2mm] \dfrac{2(L+1)}{3(L+2)} & \text{regime 2.} \end{cases} \tag{25}$$

The correlation length exponents are seen to satisfy the general scaling relation $2\nu_h = 1 + 1/\delta$, which follows from the relation

$$f_s\xi^2 \sim \text{constant} \tag{26}$$

where $f_s \sim p^{1+1/\delta}$ is the singular part of the bulk free energy and the exponents δ are those following from the singular behaviour of (16) [3–5]†.

The magnetic Ising case at $\lambda = 5\pi/16$ is of particular interest. From (16) we find

$$f_s \sim 4\sqrt{3}\,\frac{\sin(\pi/5)}{\cos(\pi/30)}\, p^{16/15} \qquad \text{as } p \to 0. \tag{27}$$

On the other hand, from (23) we have

$$\xi \sim \frac{1}{8\sqrt{3}\sin(\pi/5)}\, p^{-8/15} \qquad \text{as } p \to 0. \tag{28}$$

Combining these results gives the universal magnetic Ising amplitude

$$f_s\xi^2 = \frac{1}{16\sqrt{3}\sin(\pi/5)\cos(\pi/30)} = 0.061\,728\,589\ldots \qquad \text{as } p \to 0. \tag{29}$$

† The same correlation length exponents should hold for L even, for which the integrable perturbation is thermal-like. The scaling relation is now $2\nu_t = 2 - \alpha$, where ν_t and α are as given in (25) and [3–5], respectively. In particular, (25) gives the Ising value $\nu_t = 1$ for $L = 2$ in regime 1, as expected.

This is in precise agreement with the field-theoretic result obtained recently by Delfino and Mussardo, starting from Zamolodchikov's S-matrix and using the form-factor bootstrap approach [17, 18]. Full details of our calculations will be given elsewhere.

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